On The Number of Independent Channels in Multi-antenna Systems

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***Abstract*—In multi-antenna systems the use of multiple antennas at one end or both ends of the link produces multiple channels. A useful, although ill-defined, metric for such a link is the number of independent channels provided. In this paper, we discuss several candidate metrics and compare their utility in the Rayleigh fading, single-input multiple-output case. We show that most of the metrics available in the literature have limitations and can exhibit non-physical behaviour. In order to improve on their performance, we develop two novel measures for the number of independent channels based on the statistical construction of the channel and channel capacity. These two measures are then extended to multiple-input multiple-output systems, Rician channels and arbitrary channel models.**

***Index Terms*—Channel characterization, MIMO systems.**

I. Introduction

CONSIDER an M-element single-input multiple-output (SIMO) system. For such a system, performance measures such as capacity, bit-error-rate (BER) or signal-to-noise ratio (SNR) depend on the nature of the M channels from transmitter (TX) to receiver (RX). It is well-known that correlation between the channels usually reduces performance. The effects of correlation can be largely characterised by the $M \times M$ correlation matrix, but a more pragmatic approach would be to develop a single measure of the *overall correlation* as in [1]. Instead of measuring this overall correlation, we could approach the problem from the opposite viewpoint and ask, how many independent channels are there? The concept of the number of independent channels (NIC) has obvious importance, both as a practical measure and as a source of theoretical insight into channel behaviour. Hence, in this paper we consider this fundamental question. In [2]–[8], related work has appeared which directly or indirectly leads to measures of NIC. Also, in [9]–[11], Lozano *et al.* have developed a detailed investigation of single-user MIMO capacity for a variety of scenarios, including Ricean, correlated Rayleigh, polarized and interference channels. In this context, considerable insight is drawn from considering capacity slopes and power offset values. Hence, these concepts are also considered here.

To motivate our approach, consider two widely spaced receive antennas in a rich scattering environment. Here, we are comfortable in accepting that a single transmitter has two independent channels on which to communicate. If the two antennas are co-located then there is only one channel. Hence, for small, non-zero separations we expect to encounter a value of NIC such that $1 < \text{NIC} < 2$. Several metrics have been proposed to measure NIC [2]–[8] or similar concepts in the context of a SIMO system. Several previously defined metrics are ad-hoc in nature [2]–[5], some are inherently insensitive to the correlations between antennas [6], [7] and some have unreasonable physical properties. Hence, this apparently simple situation is largely unresolved.

To begin, we enumerate desirable properties of a NIC metric in a SIMO system. These are:

1) $1 \leq \text{NIC} \leq M$;
2) the metric should be simple to compute and have simple properties;
3) the metric should measure a physical quantity or have physical meaning;
4) the metric should make physical sense in a range of channel conditions.

Since NIC is not a single, universally understood quantity, there will be debate over which metric is best. For example, in this paper we present two new metrics. One is purely a function of the channel and the second is capacity based. Since capacity is inherently dependent on SNR it can be argued that capacity cannot be used to measure NIC since this is independent of the noise at the receiver. However, it can also be argued that at different SNRs, different degrees of freedom in the system become available to act as independent channels. Also, capacity is a well-known concept and its use is somewhat intuitive in this context. Another interesting issue is whether NIC should increase as more antennas are deployed. Clearly, if an antenna array is fixed (in the sense that both the antenna positions and the length are fixed) and an additional antenna is added then NIC should increase. However, if an extra antenna is added and the antennas are rearranged, then there is no guarantee that NIC will increase. This is made clear from a simple example. Consider an equally spaced array of 3 antennas, labeled, from left to right, 1, 2 and 3. Now assume a fourth antenna is added but the 4 antennas are rearranged so that antennas 1 and 2 and antennas 3 and

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4 are co-located in the positions of the original antennas 1 and 3. In this scenario, NIC must drop. It is clear that the behaviour of NIC is somewhat subtle and this perhaps explains why no simple measure is widely accepted at present. Hence, we adopt a rather broad definition of an independent channel as a degree of freedom in the system which helps to increase SNR or rate. In our evaluation of various NIC measures, we consider a linear array with a fixed length of one wavelength and equal spacing between the antennas. For large numbers of antennas, the array becomes densely packed, providing a more rigorous test of whether the metrics provide sensible results. In addition, in such scenarios, mutual coupling becomes important and this is also considered.

The rest of the paper is laid out as below. Section II details the SIMO system and channel model. Section III summarizes previous NIC metrics and Sec. IV develops two new metrics. In Sec. V, SIMO results are extended to MIMO systems, line-of-sight (LOS) channels and general channel models. Section VI gives results and conclusions appear in Sec. VII.

II. SIMO System

For concreteness, consider a SIMO system where the $M \times 1$ channel vector, $\mathbf{h}$, is complex Gaussian with a zero mean vector and covariance matrix, $\mathbf{R}$. Notationally, we describe $\mathbf{h}$ as $\mathcal{CN}(\mathbf{0},\mathbf{R})$. Hence, we are considering correlated Rayleigh fading. This assumption is generalized later, but since it is the most important special case, a thorough investigation is warranted.

Assuming the array is not so widely spaced as to encounter different path loss or shadowing, we can assume, without loss of generality, that the diagonal elements of $\mathbf{R}$ are unity, and hence, $\mathrm{tr}(\mathbf{R}) = M$. The elements of $\mathbf{R}$ are denoted by $(\mathbf{R})_{ij} = r_{ij}$. There are a very large number of models for the correlation structure in a SIMO system. Here we consider four types.

1) Exponential correlation: $r_{ij} = a^{-d_{ij}}$, where d_{ij} is the distance between antennas i and j and $0 < a < 1$ [12].
2) The Jakes model: $r_{ij} = J_0(2\pi a d_{ij})$, where $J_0(.)$ is the zero-th order Bessel function [13].
3) A Gaussian decay model: $r_{ij} = a^{-d_{ij}^2}$ [10], [14].
4) A square root model: $r_{ij} = a^{-\sqrt{d_{ij}}}$.

These models, parameterized by the arbitrary parameter, a, cover a range of scenarios from continuous decay (exponential) to oscillation (the Bessel function). Sample curves are shown in Fig. 1, where Expon, Jakes, Gauss and Sqrt represent correlation types 1) - 4) respectively. The correlation models are chosen for simplicity and coverage of a range of behaviours rather than physical reality. All models shown in Fig. 1 are parameterized by a, so that the correlation drops to 0.5 at a spacing of half a wavelength. The spacing at which the correlation is 0.5 is denoted the "decorrelation distance," or $d_{0.5}$, and can be used to set the value of a. It is convenient to write the correlated channel vector, $\mathbf{h}$, in terms of an independent and identically distributed (iid) $\mathcal{CN}(\mathbf{0},\mathbf{I}_M)$ vector $\mathbf{u}$, where $\mathbf{I}_M$ is the $M \times M$ identity matrix. Hence, we have

$$\mathbf{h} = \mathbf{A}\mathbf{u}, \tag{1}$$

where $\mathbf{A}$ is a lower triangular $M \times M$ matrix obtained from the Cholesky decomposition, $\mathbf{R} = \mathbf{A}\mathbf{A}^\dagger$, and † denotes the Hermitian transpose. Two special cases are of particular interest. If the antennas in the array are all co-located, then $\mathbf{R}$ is a matrix of ones, $\mathbf{R} = \mathbf{1}_M$, and NIC should be unity. For a widely separated array, $\mathbf{R} = \mathbf{I}_M$ and NIC should be M.

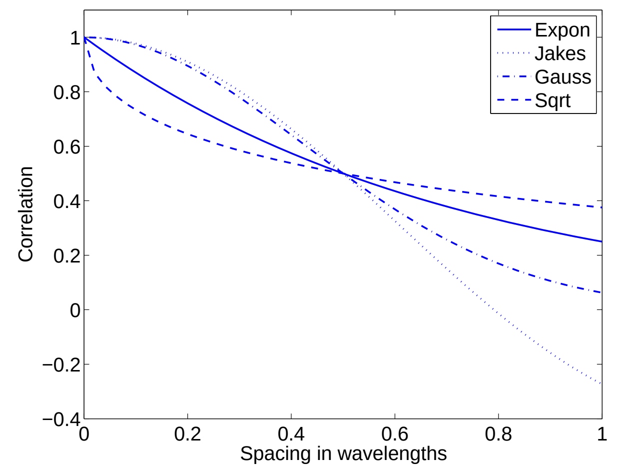

Fig. 1. The four correlation models parameterized by a decorrelation distance of 0.5λ.

III. NIC Metrics

A. Ad-hoc metrics

In [5], a metric was suggested to capture correlation structure and measure the "power balance and degrees of freedom." The metric is defined by

$$\mathrm{NIC}_1 = \frac{\mathrm{tr}(\mathbf{R})}{\lambda_{\max}(\mathbf{R})} = \frac{M}{\lambda_{\max}(\mathbf{R})}, \tag{2}$$

where $\mathrm{tr}(\cdot)$ denotes trace and $\lambda_{\max}(\cdot)$ is the maximum eigenvalue. In a similar vein, an "effective degrees of freedom" metric was used in [3], defined by

$$\mathrm{NIC}_2 = \frac{[\mathrm{tr}(\mathbf{R})]^2}{\mathrm{tr}(\mathbf{R}^2)} = \frac{M^2}{\mathrm{tr}(\mathbf{R}^2)}. \tag{3}$$

Note that NIC_2 is a one-to-one function of $\zeta(\mathbf{R}) = \mathrm{tr}(\mathbf{R}^2)/M$, the dispersion of $\mathbf{R}$ or correlation number [9], given by $\mathrm{NIC}_2 = M/\zeta(\mathbf{R})$. Both metrics satisfy $\mathrm{NIC} = 1$ for a co-located array and $\mathrm{NIC} = M$ for a widely separated array.

B. Capacity based metrics

In a study of MIMO channels [4], the number of spatial degrees of freedom is defined as

$$\mathrm{DOF} = \lim_{\rho\to\infty} \frac{C(\rho)}{\log \rho}, \tag{4}$$

where $C(\rho)$ is the capacity at SNR ρ and $C(\rho)$ is fixed as fading is not considered. A version of this definition can also be used in the fading SIMO context. In [6], a general study of DOF and diversity (DIV) for SIMO and MIMO systems is undertaken. The resulting definitions are also in terms of high SNR and in this limiting regime, $\rho \to \infty$, neither DOF or DIV are related to $\mathbf{R}$. Hence, the DOF and DIV values are independent of correlation. In fact, DOF defined in (4) is not a measure of the number of independent channels but a quantity related to the number of parallel channels in the multiplexing perspective. Hence, for a SIMO system its value is always unitary, irrespective of the number of antennas and the correlation matrix $\mathbf{R}$. Clearly this is not what is required here, where it is exactly the effects of correlation on the channels that is of interest. Hence we propose non-limiting versions of DOF [4], [6] as

$$\text{NIC}_3 = \frac{\mathbb{E}[C(\rho, M)]}{\log \rho}, \tag{5}$$

$$\text{NIC}_4 = \frac{\mathbb{E}[C(\rho, M)]}{\mathbb{E}[C(\rho, 1)]}, \tag{6}$$

where $C(\rho, n)$ is the SIMO capacity with n antennas and the SNR is ρ.

The values of the mean capacity for a spatially correlated SIMO system are given by [15]

$$\mathbb{E}[C(\rho, M)] = \frac{1}{\log 2} \sum_{j=1}^{M} b_j \exp\left(\frac{1}{\rho\lambda_j}\right) E_1\left(\frac{1}{\rho\lambda_j}\right), \tag{7}$$

where $\lambda_1, \lambda_2, \ldots, \lambda_M$ are the eigenvalues of $\mathbf{R}$, $E_1(\cdot)$ is the exponential integral and $b_j = \lambda_j^{M-1} \prod_{r \neq j} (\lambda_j - \lambda_r)^{-1}$. Note that the result in (7) is only valid when the matrix $\mathbf{R}$ has unequal eigenvalues. In the two special cases of interest where $\mathbf{R} = \mathbf{I}_M$ or $\mathbf{R} = \mathbf{1}_M$ we have the respective special cases [1], [15]

$$\mathbb{E}[C(\rho, M)] = \frac{1}{M} \sum_{k=0}^{M-1} \frac{\exp(1/\rho)}{k!\rho^k} \Bigg\{ (-1)^k E_1\left(\frac{1}{\rho}\right) + \sum_{r=1}^{k} (-1)^{k-r} \binom{k}{r} \rho^r \Gamma\left(r, \frac{1}{\rho}\right) \Bigg\}, \tag{8}$$

$$\mathbb{E}[C(\rho, M)] = \exp(1/M\rho) E_1\left(\frac{1}{M\rho}\right), \tag{9}$$

where $\Gamma(\cdot, \cdot)$ is the upper incomplete gamma function. The result for $\mathbf{R} = \mathbf{I}_M$ is given in [15] and the result for $\mathbf{R} = \mathbf{1}_M$ is in [1].

Substituting (8) and (9) in (5) and (6) shows that neither NIC_3 nor NIC_4 varies between 1 and M as is the case for the first two metrics. It is easy to prove that in a SIMO system with Rayleigh fading, NIC_3 and NIC_4 converge to unity when the SNR tends to infinity. Although for low to moderate SNR values, NIC_3 and NIC_4 increase with the number of antennas M, such dependence cannot be clearly interpreted as will be confirmed by numerical results in Section VI.

In [9]–[11], Lozano *et al.* develop two important metrics which reflect the distinctiveness of each channel [10] and reveal the impact of correlation. At high SNR [10], [11], the power offset, $\mathcal{L}_\infty$, is defined with respect to a reference, unfaded and orthogonal channel. At low SNR [9], the capacity slope, S_0, is used. Both parameters quantify "the capacity impact of an entire correlation matrix" [9]. Hence, although neither metric aims to measure NIC, their rigorous basis and relationship to the correlation structure makes them attractive candidates. The power offset, $\mathcal{L}_\infty$, is a moderately complex metric and its value is tied to an analytical derivation of capacity. Hence, extensions to general channel models are difficult. For these reasons we do not pursue this metric further.

The capacity slope, S_0, is simpler and has a finite range where the extremes are achieved by iid or fully correlated channels. In this sense, it is similar to the simple metrics in (2) - (3). Computing S_0 for the correlated Rayleigh SIMO channel gives

$$S_0 = \frac{2M^2}{M^2 + \text{tr}(\mathbf{R^2})}.$$

In order to scale S_0, so that it runs from M to 1 as the correlation increases from an iid channel to a fully correlated channel, we define the metric

$$\text{NIC}_{S_0} = (M+1)S_0 - M = \frac{2(M+1)M^2}{M^2 + \text{tr}(\mathbf{R^2})} - M,$$

which is related to NIC_2 via the result $\text{NIC}_{S_0} = ((M+2)\text{NIC}_2 - M)/(1 + \text{NIC}_2)$.

C. Receiver based metrics

In [8], it was proposed to think of diversity as equivalent to the shape parameter of the combiner output in a maximum ratio combiner (MRC). In MRC, the output SNR is given by $\gamma = \mathbf{h}^\dagger\mathbf{h} = \mathbf{u}^\dagger\mathbf{A}^\dagger\mathbf{A}\mathbf{u}$, using (1). Here, it is assumed without loss of generality that the transmit SNR is unity. Since the eigenvalues of $\mathbf{A}^\dagger\mathbf{A}$ are the same as the eigenvalues of $\mathbf{R} = \mathbf{A}\mathbf{A}^\dagger$, we can also write $\gamma = \mathbf{u}^\dagger\mathbf{\Phi}^\dagger\mathbf{\Lambda}\mathbf{\Phi}\mathbf{u} = \tilde{\mathbf{u}}^\dagger\mathbf{\Lambda}\tilde{\mathbf{u}} = \tilde{\mathbf{u}}^\dagger\text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_M)\tilde{\mathbf{u}}$ where $\tilde{\mathbf{u}}$ is an iid, $M \times 1$, $\mathcal{CN}(\mathbf{0}, \mathbf{I}_M)$ vector, $\mathbf{R} = \mathbf{\Phi}^\dagger\mathbf{\Lambda}\mathbf{\Phi}$ and diag$(\cdot)$ represents a diagonal matrix. Finally, we have the result

$$\gamma = \sum_{i=1}^{M} \lambda_i |\tilde{u}_i|^2. \tag{10}$$

In an iid fading channel, $\lambda_1 = \lambda_2 = \cdots = \lambda_M = 1$ and γ has a Chi-squared distribution with shape parameter M. In [8], it is suggested that in the correlated fading case also, a gamma approximation to γ will have a corresponding shape parameter which can be interpreted as a measure of diversity. Applying this idea here, we note that

$$\mathbb{E}[\gamma] = \sum_{i=1}^{M} \lambda_i, \ \text{Var}[\gamma] = \sum_{i=1}^{M} \lambda_i^2. \tag{11}$$

The standard method of moments [16] approach to fitting the gamma shape parameter is to set the shape parameter equal to the ratio of $\mathbb{E}[\gamma]^2$ and Var(γ). Hence, we have a measure of diversity, or NIC, given by

$$\begin{aligned} \text{NIC} &= \frac{\mathbb{E}[\gamma]^2}{\text{Var}[\gamma]} = \frac{[\sum_{i=1}^{M} \lambda_i]^2}{\sum_{i=1}^{M} \lambda_i^2} \\ &= \frac{\text{tr}(\mathbf{R})^2}{\text{tr}(\mathbf{R}^2)} = \frac{M^2}{\text{tr}(\mathbf{R}^2)}, \end{aligned} \tag{12}$$

which is the same as NIC_2. Hence the ad-hoc metric, NIC_2, has justification from this viewpoint. This metric has also been proposed in [17] as the indicator of the diversity order, whose physical interpretation is the reduction of the SNR variance due to the use of multiple antennas with respect to a SISO system.

IV. Proposed Metrics

A. Construction metric

In equation (1), $\mathbf{h} = \mathbf{A}\mathbf{u}$, where $\mathbf{A}$ is lower triangular. Hence, the iid vector $\mathbf{u}$ can be thought of as the input consisting of M independent channels. The first two elements of $\mathbf{h}$ are

$$h_1 = A_{11}u_1, \tag{13}$$
$$h_2 = A_{21}u_1 + A_{22}u_2. \tag{14}$$

Hence, h_1 contains an independent channel of power $|A_{11}|^2$. The second element, h_2, contains part of the first channel, the $A_{21}u_1$ term, and an independent component of power $|A_{22}|^2$. Similarly,

$$h_r = \sum_{i=1}^{r-1} A_{ri}u_i + A_{rr}u_r, \tag{15}$$

and the $r-$th element has an independent component of power $|A_{rr}|^2$. In total, the vector $\mathbf{h}$ contains independent components of total power $\sum_{r=1}^{M} |A_{rr}|^2$. This is the proposed novel NIC metric

$$\text{NIC}_{\text{CON}} = \sum_{r=1}^{M} |A_{rr}|^2. \tag{16}$$

This has a physical interpretation as the total power of the independent components contained in $\mathbf{h}$ and has the potential advantage of being independent of SNR. The drawback of this approach is that it only strictly applies to Gaussian channels where the scattered components have the form given in (1). Nevertheless, it can be extended to other channels as shown in Sec. V.

Since the Cholesky decomposition of $\mathbf{R}$ depends on the ordering of $\mathbf{h}$, different orderings of $\mathbf{h}$ give different values of NIC. At first sight, this non-uniqueness appears to be a problem. However, for physical reasons only one ordering (or set of equivalent orderings) is reasonable and the non-uniqueness problem disappears. To understand this property, consider the fact that (13) - (15) themselves imply an ordering. Equations (13) and (14), for example, construct channel 2 after channel 1. Hence, whichever antenna is deemed to correspond to h_1 is considered first and the antenna corresponding to h_2 is considered second. For the metric to be physically reasonable it must satisfy certain properties. For example, NIC must increase if an extra antenna is added without any rearrangement of the existing antennas[1]. Consider an array where antennas are to be placed at positions 1, 2 and 3. Positions 1 and 3 locate the ends of the array and position 2 is in the centre. Let NIC(i,j,k) denote the NIC value with antennas at positions i, j and k. Here, we must have NIC$(1,2,3) >$ NIC$(1,3)$. The way to satisfy this constraint is to use the ordering 1,3,2 or 3,1,2. By symmetry these are equivalent. Also, since positions 1 and 3 are occupied first, the addition of location 2, using (15), adds an extra contribution to the existing NIC value. Hence, instead of 3! possible orderings giving different values of NIC, we have 2 possible equivalent orderings giving a unique answer. Similarly, with 4 antennas and 5 antennas, we have the possible orderings 1,4,2,3 and 1,5,3,2,4 respectively using a similar labeling. This natural construction ordering is also the ordering that maximizes NIC. This way of viewing the ordering also ensures that NIC increases when extra antennas are added without rearrangement.

[1]Here, we assume a fixed array length and fixed positions for the original antennas so that the new antenna must be placed between two of the original antennas.

As for metrics 1-2, the new metric ranges from 1 to M. A simple lower bound can be found for the construction metric in the specific case of the exponential correlation model. In this scenario, $r_{ij} = a^{-d_{ij}}$, and in the case of a uniformly spaced array, $r_{ij} = a^{-d|i-j|}$, where $d = d_{12}$. For this correlation matrix, it can be shown that the Cholesky decomposition has leading diagonal given by $(1, \sqrt{1-\alpha^2}, \sqrt{1-\alpha^2}, \ldots, \sqrt{1-\alpha^2})$, where $\alpha = a^{-d}$. Proof is straightforward by induction and is given in Appendix A. Hence we have the lower bound

$$\text{NIC}_{\text{CON}} \geq 1 + (M-1)(1-\alpha^2) = M - (M-1)\alpha^2. \tag{17}$$

This bound satisfies $\text{NIC}_{\text{CON}} = 1$ for $\alpha = 1$ and $\text{NIC}_{\text{CON}} = M$ for $\alpha = 0$. However, it is a lower bound since the original ordering of the antennas is used, rather than the optimal, construction ordering. Nevertheless, (17) provides a useful rule of thumb. In situations with an exponential correlation structure the number of channels lost due to correlation is at most $(M-1)\alpha^2$. Note that α^2 is the proportion of the channel variability at antenna i that is predictable based on a knowledge of channel $i-1$ [16]. Hence, $(M-1)\alpha^2$ has a simple interpretation as the maximum number of channels that can be lost scaled by the fraction of any channel that is known conditional on a neighbouring channel. Since at most $(M-1)\alpha^2$ channels are lost, we can reverse the above argument to conclude that $\alpha \geq \sqrt{\frac{\kappa}{M-1}}$ is required to lose κ channels.

B. Capacity metric

We now propose a second metric, based on the capacity of the correlated system under consideration. Let the capacity of an M antenna SIMO system with correlation $\mathbf{R}$ and SNR ρ be $C(\rho, M)$ [18]. We define NIC_{CAP} by the number of receive antennas in an uncorrelated system of equal ergodic capacity and equal SNR ρ. Since $C(\rho, M)$ will not necessarily correspond to an iid SIMO system with an integer number of antennas, we have

$$\mathbb{E}[C_{\text{iid}}(\rho, m)] \leq \mathbb{E}[C(\rho, M)] \leq \mathbb{E}[C_{\text{iid}}(\rho, m+1)], \tag{18}$$

where the capacity of an m antenna iid SIMO system is denoted $C_{\text{iid}}(\rho, m)$. To determine NIC, we linearly interpolate between the two iid systems, which has the physical interpretation of switching between m and $m+1$ antennas. Since this approach is solely a function of capacity, it can be computed for any channel either by analysis (where possible) or simulation.

A drawback of any capacity based metric is that capacity increases with an extra antenna even if the new channel gain is identical to an existing channel gain (ie., the new antenna is located in such close proximity to an existing antenna that the two channels are identical). Hence, capacity is measuring array gain as well as the impact of independent channels. In order to account only for the contribution of independent channels, we apply a correction factor to remove the array gain contribution. The corrected mean iid and correlated capacities are given by

$$\begin{aligned} \mu_{\text{iid}}(\rho, m) &= \mathbb{E}[C_{\text{iid}}(\rho, m)] - \Delta_m, \\ \mu(\rho, M) &= \mathbb{E}[C(\rho, M)] - \Delta_M, \end{aligned} \quad (19)$$

where the correction factor, Δ_j, for j antennas and SNR ρ can be shown from (9) to be

$$\Delta_j = \exp\left(\frac{1}{\rho j}\right) E_1\left(\frac{1}{\rho j}\right) - \exp\left(\frac{1}{\rho}\right) E_1\left(\frac{1}{\rho}\right). \quad (20)$$

The factor Δ_j can be identified as the growth in capacity over a single antenna system solely due to adding extra, identical channels. For a single antenna system, $\Delta_1 = 0$ as expected. With this notation, if $\mu_{\text{iid}}(\rho, m) \le \mu(\rho, M) \le \mu_{\text{iid}}(\rho, m+1)$, then we define the metric, NIC_{CAP}, as

$$\text{NIC}_{\text{CAP}} = m + \frac{\mu(\rho, M) - \mu_{\text{iid}}(\rho, m)}{\mu_{\text{iid}}(\rho, m+1) - \mu_{\text{iid}}(\rho, m)}. \quad (21)$$

The version of (21) which includes array gains is obtained by using $\Delta_j = 0$.

V. Extensions to MIMO Systems and non-Rayleigh Channels

A. Rician Channels

Unlike NIC_{CAP}, which is applicable to general SIMO channels, NIC_{CON} was developed for Rayleigh channels. We now extend this approach to SIMO Rician channels [19], where

$$\mathbf{h}_{RICE} = \sqrt{\frac{K}{K+1}}\mathbf{h}_{LOS} + \sqrt{\frac{1}{K+1}}\mathbf{h},$$

where K is the Rician K-factor, $\mathbf{h}$ is a correlated Rayleigh fading channel as before and $\mathbf{h}_{LOS}$ is the line-of-sight (LOS) component. The standard model for the LOS component is $\mathbf{h}_{LOS} = (1 \;\; e^{j\theta} \;\; \ldots e^{j(M-1)\theta})/\sqrt{M}$ [19] which satisfies $||\mathbf{h}_{LOS}||^2 = 1$. For this model, we propose

$$\text{NIC}_{\text{RICE}} = \frac{K}{K+1} + \frac{1}{K+1}\text{NIC}_{\text{CON}}. \quad (22)$$

Equation (22) is built on the idea that the LOS component, $\mathbf{h}_{LOS}$, is a single channel of power $\frac{K}{K+1}$ and the scattered component, $\mathbf{h}$, has NIC_{CON} channels of power $\frac{1}{K+1}$. Note that $\mathbf{h}_{LOS}$ must contribute a single channel since any perfectly correlated vector corresponds to one channel.

The properties of NIC_{RICE} are consistent with expectation. For example, as $K \to \infty$, $\text{NIC}_{\text{RICE}} \to 1$ and as $K \to 0$, $\text{NIC}_{\text{RICE}} \to \text{NIC}_{\text{CON}}$. Also, as the scattered component becomes perfectly correlated, $\text{NIC}_{\text{CON}} \to 1$ and $\text{NIC}_{\text{RICE}} \to 1$. All these special cases agree with the desirable properties of any NIC measure. Finally, it is of interest to consider the iid Rician channel where $\text{NIC}_{\text{CON}} = M$ and the corresponding value of NIC_{RICE} is $\frac{K+M}{K+1}$. This indicates a loss of $M - (K+M)/(K+1) = \frac{K}{K+1}(M-1)$ channels due to the presence of the LOS component. This is intuitively sensible since there are a maximum of $M-1$ channels that can be lost and $\frac{K}{K+1}$ is the fraction of the channel that is LOS. Hence, the number of channels lost is the maximum number scaled by the fraction of the channel that is LOS.

B. MIMO Links

In the Rayleigh fading MIMO case, we consider the classic Kronecker model [20] where the $M \times N$ channel matrix is defined by $\mathbf{H} = \mathbf{R}_r^{\frac{1}{2}}\mathbf{U}\mathbf{R}_t^{\frac{1}{2}}$. In this formulation, there are M receive antennas, N transmit antennas, $\mathbf{R}_r$ and $\mathbf{R}_t$ are the receive end and transmit end correlation matrices and $\mathbf{U}$ is an $M \times N$ iid Rayleigh fading channel matrix. In the MIMO case there are at most MN independent channels and so $1 \le \text{NIC} \le MN$ is a desirable property.

Using the construction approach to NIC, we have $\text{NIC}_{\text{CON,MIMO}} = \sum_{j=1}^{MN} |A_{jj}|^2$ where $\mathbf{R} = \mathbf{A}\mathbf{A}^\dagger$ and $\mathbf{R}$ is the correlation matrix, $\mathbf{R} = \mathbb{E}[\text{vec}(\mathbf{H})\text{vec}(\mathbf{H})^\dagger]$. In Appendix B we show that if $\mathbf{R}_r = \mathbf{A}^{(r)}\mathbf{A}^{(r)\dagger}$ and $\mathbf{R}_t = \mathbf{A}^{(t)}\mathbf{A}^{(t)\dagger}$ then

$$\text{NIC}_{\text{CON,MIMO}} = \sum_{j=1}^{M} |A_{jj}^{(r)}|^2 \times \sum_{k=1}^{N} |A_{kk}^{(t)}|^2. \quad (23)$$

The right hand side of (23) is the product of the NIC_{CON} values for the corresponding SIMO and MISO systems. This result makes perfect sense. Since the Kronecker model separates the effect of correlation at the transmit and receive ends, it follows that NIC will also be separable. Equation (23) states that the NIC for a MIMO system is the NIC value at the receiver for a $1 \times M$ SIMO system multiplied by the NIC value at the transmitter for a $N \times 1$ MISO system. The simplicity and intuition of this result makes it especially appealing and we rewrite (23) as

$$\text{NIC}_{\text{CON,MIMO}}(M, N) = \text{NIC}_{\text{CON}}(M, 1) \times \text{NIC}_{\text{CON}}(1, N). \quad (24)$$

Since $1 \le \text{NIC}_{\text{CON,MIMO}}(M, 1) \le M$ and $1 \le \text{NIC}_{\text{CON,MIMO}}(1, N) \le N$ it automatically follows that $1 \le \text{NIC}_{\text{CON,MIMO}}(M, N) \le MN$ as desired. Also, as correlations at both ends tend to 0 or 1, $\text{NIC}_{\text{CON,MIMO}}(M, 1)$ tends to M or 1 and $\text{NIC}_{\text{CON,MIMO}}(1, N)$ tends to N or 1 as expected.

Overall, the MIMO extension to NIC_{CON} has a simple intuitive structure and follows the desired properties of a NIC measure. Note that for non-Kronecker models, (23) is not relevant and the original definition, $\text{NIC}_{\text{CON,MIMO}} = \sum_{j=1}^{MN} |A_{jj}|^2$, is used.

Extending NIC_{CAP} is more problematic. In the SIMO case, the NIC was set by matching the mean capacity. In the MIMO case there are now two dimensions and finding NIC at each end requires two equations. The obvious approach is to match both the capacity mean and variance, requiring a two-dimensional interpolation extension of (21). Interpolating in the two-dimensional (M, N) space for mean capacity gives a curve of possible system dimensions rather than the single point found in the SIMO case. Hence, we obtain a curve for mean capacity and a curve for capacity variance and hope to encounter a unique intersection point providing a unique NIC value. This approach requires extremely accurate knowledge of the capacity moments or the interpolation is not stable. This level of accuracy can be obtained from analysis in some limited cases (mainly for Rayleigh channels) or by very long simulation runs. This approach is no longer simple to compute which makes it less desirable as a NIC measure. In practice there is a more significant problem. Numerical tests show that in some cases a solution does not exist in the region of interest.

An example of this is shown in Fig. 9 and is discussed further in Sec. VI. As a result of these issues, a capacity based NIC measure is not proposed for the MIMO case.

C. General Channel Models

Consider the general $M \times N$ MIMO channel defined by $\mathbf{H} = \mu\mathbf{M} + \sqrt{1-\mu^2}\mathbf{H}_0$, where the normalized channel, $\mathbf{H}$, satisfies $E[|H_{ij}|^2] = 1$ and $0 \leq \mu \leq 1$. Similarly, $\mathbf{H}_0$ is normalized so that $E[|(\mathbf{H}_0)_{ij}|^2] = 1$. The matrix, $\mu\mathbf{M}$, is the possibly non-zero mean of $\mathbf{H}$, $\mu\mathbf{M} = E[\mathbf{H}]$, and $\sqrt{1-\mu^2}\mathbf{H}_0 = \mathbf{H} - \mu\mathbf{M}$ is the zero-mean component. We assume that $|M_{ij}| = 1 \; \forall \; i, j$ since the antennas are not widely separated. The correlation matrix, $\mathbf{R}$, of $\mathbf{H}_0$ is defined as $\mathbf{R} = \mathbf{A}\mathbf{A}^\dagger$. With these definitions, the obvious extension of (22) to the general MIMO channel is

$$\text{NIC}_{\text{CON,MIMO}} = \mu^2 \text{NIC}(\mathbf{M}) + (1-\mu^2)\sum_{j=1}^{MN} |A_{jj}|^2, \quad (25)$$

where $\text{NIC}(\mathbf{M})$ is the number of independent channels in the $\mathbf{M}$ matrix.

To compute $\text{NIC}(\mathbf{M})$ for a deterministic $M \times N$ matrix, $\mathbf{M}$, we use the QR decomposition $\mathbf{M} = \mathbf{\Phi}\mathbf{B}$ where $\mathbf{\Phi}$ is an $M \times M$ orthogonal matrix satisfying $\mathbf{\Phi}^\dagger\mathbf{\Phi} = M\mathbf{I}_M$ and $\mathbf{B}$ is an $M \times N$ upper triangular matrix with $B_{11} = 1$. In an analogous manner to the construction of NIC_{CON}, we note that the QR decomposition is not unique as the order of the columns of $\mathbf{M}$ are changed. Hence, the same ordering is used for the columns of $\mathbf{M}$ as is used in the NIC_{CON} metric for SIMO channels. Using the QR decomposition of the reordered $\mathbf{M}$ we have

$$\begin{aligned} \mathbf{M}_{\cdot 1} &= B_{11}\mathbf{\Phi}_{\cdot 1}, \\ \mathbf{M}_{\cdot 2} &= B_{12}\mathbf{\Phi}_{\cdot 1} + B_{22}\mathbf{\Phi}_{\cdot 2}, \end{aligned} \quad (26)$$

where $\mathbf{M}_{\cdot i}$ and $\mathbf{\Phi}_{\cdot i}$ are the $i-$ th columns of $\mathbf{M}$ and $\mathbf{\Phi}$ respectively. Hence, as for NIC_{CON}, $|B_{11}|^2 = 1$ gives the power of the first linearly independent column in $\mathbf{M}$ and $|B_{22}|^2$ gives the power of the second linearly independent column. The total power of the linearly independent columns in $\mathbf{M}$ is therefore $\sum_{i=1}^{\min(M,N)} |B_{ii}|^2$. Since each column of $\mathbf{M}$ can only provide 1 channel (as discussed in Sec. V), it follows that $\text{NIC}(\mathbf{M}) = \sum_{i=1}^{\min(M,N)} |B_{ii}|^2$. Hence,

$$\text{NIC}_{\text{CON,MIMO}} = \mu^2 \sum_{i=1}^{\min(M,N)} |B_{ii}|^2 + (1-\mu^2)\sum_{j=1}^{MN} |A_{jj}|^2. \quad (27)$$

D. Mutual Coupling

In all the work up to this point, simple models for spatial correlation only have been considered. The NIC metrics have been evaluated based on logical arguments such as the statement that NIC cannot decrease if an extra antenna is added to a fixed array. Such statements are reasonable when only spatial correlation is considered and we ignore the physical effects of the antennas. To include such physical effects, we now consider mutual coupling. In [21] and [22] a commonly used, simple model is described for mutual coupling where the effects of coupling are modeled by coupling matrices at the transmitter or receiver or both. In this paper, we use exactly the same model as that given in [22] with half wavelength wire dipoles. If the coupling matrix is denoted $\mathbf{C}$ in a SIMO system then the new correlation matrix for the system is simply $\mathbf{CRC}^\dagger$. Results for this scenario are given in Sec. VI. Note that coupling not only affects the correlation but also the received power [21]. Hence, in some sense the capacity metric, NIC_{CAP}, has an advantage over NIC_{CON} as it explicitly varies with SNR whereas NIC_{CON} is solely a function of the correlations.

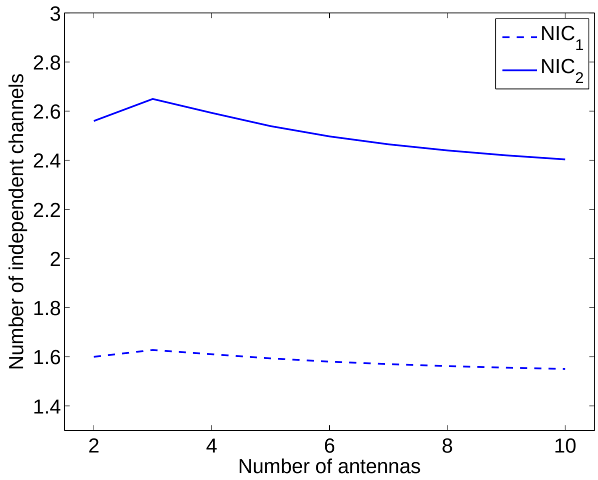

Fig. 2. NIC vs antenna numbers for metrics 1 and 2 with the exponential correlation model and $d_{0.5} = 0.5\lambda$.

VI. Discussion and Results

Under normal conditions, say an array with half wavelength spacings, the different metrics can all behave reasonably and it is difficult to argue their relative merits. Hence, we consider the realistic case where increasing numbers of antennas may be employed but the array length is fixed. In particular, we consider a linear array with a length of one wavelength and equal spacing between the antennas. For large numbers of antennas, the array becomes densely packed, providing a more rigorous test of whether the metrics provide sensible results. This situation is interesting in its own right and there has been work on the capacity of such arrays [23]–[26]. In the first results, we ignore coupling effects [21], [22], [25], [26] since the focus is on statistical channel models. However, in Fig.10 we add the effects of coupling using the commonly used, simple model in [22].

Results for the fixed array length scenario are shown in Figs. 2 to 8. Fig. 2 shows that both ad-hoc metrics, NIC_1 - NIC_2, reach a peak value for NIC and then decay away in the case where the correlation is exponential. Although not shown, this decay is even more noticeable for the Gaussian and Jakes correlation models. It is difficult to argue that NIC decreases steadily after a few antennas are deployed. Hence, these metrics are not recommended.

Fig. 3 shows the SNR sensitivity of the capacity based metrics. For NIC_4, NIC varies from 1.8 to 4.8 as the SNR moves from 0 dB to 20 dB with 20 antennas. This is a large

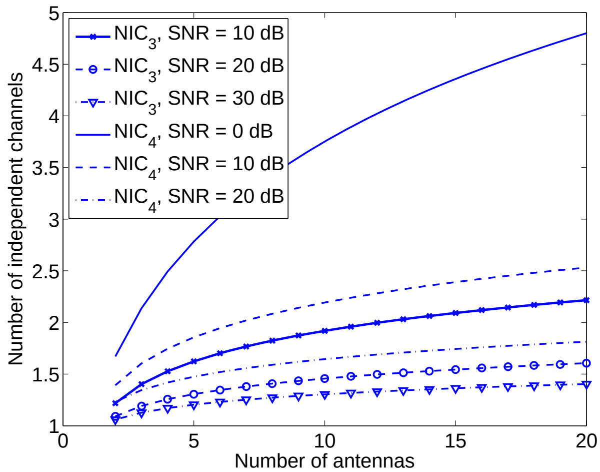

Fig. 3. NIC vs antenna numbers for metrics 3 and 4 with the exponential correlation model and $d_{0.5} = 0.5\lambda$.

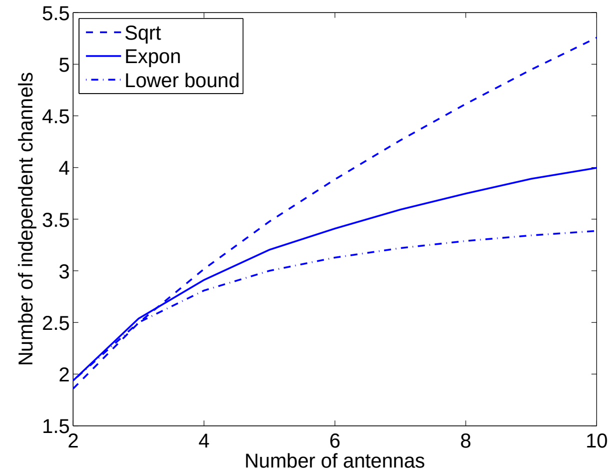

Fig. 4. NIC vs antenna numbers for NIC_{CON} with various correlation models and $d_{0.5} = 0.5\lambda$. Also shown is the lower bound for the exponential case. Array length fixed at one wavelength; no coupling considered.

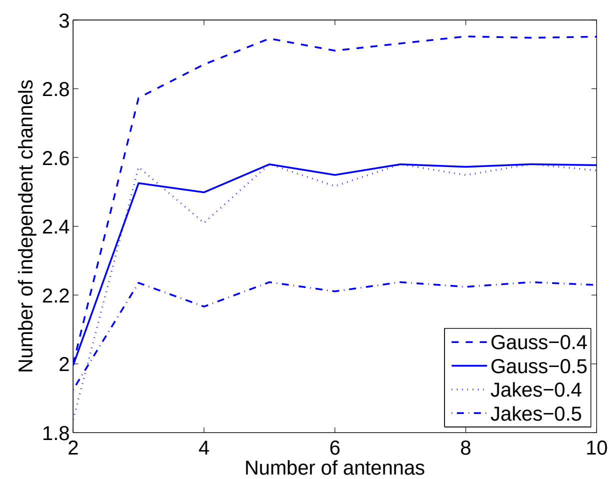

Fig. 5. NIC vs antenna numbers for NIC_{CON} with various correlation models and $d_{0.5} \in \{0.4\lambda, 0.5\lambda\}$. Array length fixed at one wavelength; no coupling considered.

sensitivity to SNR which is a potentially undesirable feature in a NIC metric. Hence, since a simple direct connection between such metrics with the number of independent channels is hard to find, NIC_3 and NIC_4 are not recommended.

Results for NIC_{S_0} show an approximate linear growth as more antennas are deployed. This is not physically plausible for very large values of M. This behaviour can be shown by considering a decorrelation distance, $d_{0.5} = 0.5\lambda$. Consider a typical $M \times M$ correlation matrix, $\mathbf{R}$, with $d_{0.5} = 0.5\lambda$ and $M = 2L$. We assume that the correlations in $\mathbf{R}$ decay from 1 to 0.5 as the spacing increases from 0 to 0.5λ. Similarly, we assume that the absolute values of the correlations remain below 0.5 for spacings between 0.5λ and 1λ. The value of NIC_{S_0} with an $\mathbf{R}$ matrix of this form should be larger than the value with the simple matrix $\mathbf{R_u}$ which contains higher correlations and is defined by

$$(\mathbf{R_u})_{ij} = \begin{cases} 1 & |i-j| \leq L-1 \\ \frac{1}{2} & |i-j| \geq L \end{cases}.$$

With this simple correlation matrix, $\text{tr}(\mathbf{R_u}^2) = (13L^2 - 3L)/4$, which gives $\text{NIC}_{S_0} \geq (6L+38)(29-3/L)^{-1}$. For larger M, L is large and NIC_{S_0} is lower bounded by $6L/29$. Hence, NIC_{S_0} increases as M increases. For this reason, NIC_{S_0} is not considered a plausible metric; the capacity slope does not appear to be directly linked to NIC.

The new construction metric is shown in Figs. 4 and 5. For the correlation models which decay rapidly close to the origin (Sqrt and Expon) the NIC metric increases steadily and begins to plateau. The simple lower bound for the exponential case is a reasonable approximation to NIC_{CON} for up to 5 antennas in the 1λ array. For the correlation models which are smooth at the origin (Gauss and Jakes) the behaviour is different. When $d_{0.5} = 0.5\lambda$, for odd numbers of antennas and for even numbers of antennas, the same rise and leveling off is observed. Moving from an even number to an odd number of antennas the NIC can drop. This is reasonable since extra antennas will not always increase NIC; it will depend on their placement. When moving from an even number to an odd number of antennas, the actual placement of all the antennas changes, with the exception of the outer antennas. Hence, this behaviour is plausible. Furthermore, these results have considerable implications for the use of statistical channel models in closely packed arrays. The behaviour of the correlations at small separations is critical and the use of different models may lead to fundamentally different behaviour. This is shown by a comparison of Figs. 4 and 5. In Fig. 4, a smooth growth in NIC occurs. Here, the correlation models have a first derivative with respect to distance that is negative at the origin. Hence the correlation functions drop away immediately at the origin. In Fig. 5, the growth behaviour is more complex, corresponding to correlation models where the derivative is zero at the origin. Hence, the correlation functions roll off more slowly from zero.

Also shown in Fig. 5 is the effect of varying the decorrelation distance with $d_{0.5} \in \{0.4\lambda, 0.5\lambda\}$. For both correlation models, NIC increases as $d_{0.5}$ decreases and the correlations are reduced. Note that an array length of 1λ is general, since results for $d_{0.5} = c\lambda$ in a 1λ array are identical to results for

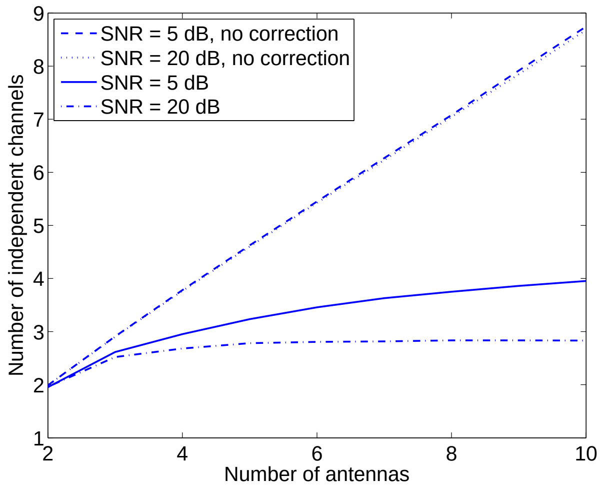

Fig. 6. NIC vs antenna numbers for $\mathrm{NIC_{CAP}}$ with the exponential correlation model. Corrected and uncorrected versions of the metric are shown at SNR $\in \{5\ \mathrm{dB}, 20\ \mathrm{dB}\}$ and $d_{0.5} = 0.5\lambda$. Array length fixed at one wavelength; no coupling considered.

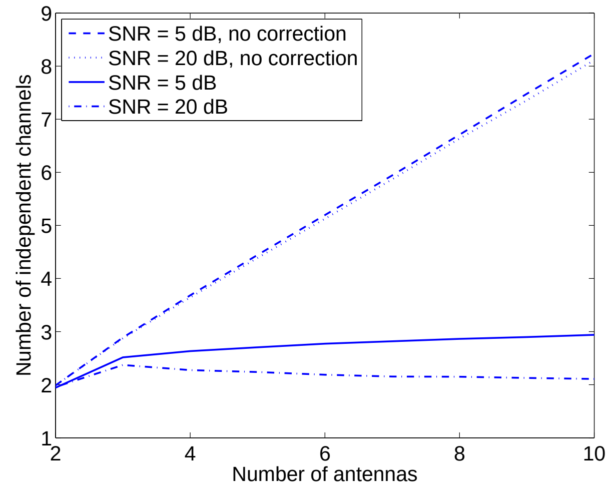

Fig. 7. NIC vs antenna numbers for $\mathrm{NIC_{CAP}}$ with the Jakes correlation model. Corrected and uncorrected versions of the metric are shown at SNR $\in \{5\ \mathrm{dB}, 20\ \mathrm{dB}\}$ and $d_{0.5} = 0.5\lambda$. Array length fixed at one wavelength; no coupling considered.

$d_{0.5} = bc\lambda$ in a $b\lambda$ array, as one is simply a scaled version of the other.

Finally, the new capacity based metric, $\mathrm{NIC_{CAP}}$, is shown for exponential and Jakes correlation models in Figs. 6 and 7, respectively. The figures were obtained for a decorrelation distance of 0.5 and $\rho = 5$ and 20 dB. Included in the figures is NIC computed without the array gain correction factor, that is using with $\Delta_m = \Delta_M = 0$ in (19). The figures clearly demonstrate the need for the correction factor. Without the correction factor, NIC increases approximately linearly for the values of M considered. A more reasonable metric is obtained with the correction factor, where the NIC reaches a steady state value of $2-3$ for the parameters considered. The correction factor removes the continual growth in NIC and also makes NIC slightly more SNR dependent and in Fig.7 we observe a slight but steady decay in NIC with an SNR of 20 dB. This fact can be explained by considering that the array gain is a dominant factor which masks the dependency on the SNR and on the correlation. Once the array gain effect is removed, the relationship between $\mathrm{NIC_{CAP}}$ and the SNR appears in both Figs. 6 and 7. For the same reason, without correction, $\mathrm{NIC_{CAP}}$ in the presence of exponential correlation (Fig. 6) and in the presence of Jakes correlation (Fig. 7) gives very similar results, showing the masking effect on the correlation.Clearly there are still some concerns over the use of capacity based NIC metrics.

In Fig. 8, results for $\mathrm{NIC_{CON}}$ and $\mathrm{NIC_{CAP}}$ are shown for $d_{0.5} = 0.5\lambda$, SNR = 5dB and the exponential and Jakes correlation models. The corrected version of $\mathrm{NIC_{CAP}}$ is used. For the exponential model, the two approaches give very similar values for NIC. However, if the $\mathrm{NIC_{CAP}}$ for 20 dB was used from Fig. 6 then the two curves would no longer be in good agreement. For the Jakes model, the two curves in Figs. 8 are not in good agreement, but closer agreement would be observed if the 20 dB curve was used from Fig. 6. It is clear that the two approaches are fundamentally different and although they may roughly coincide for certain values of

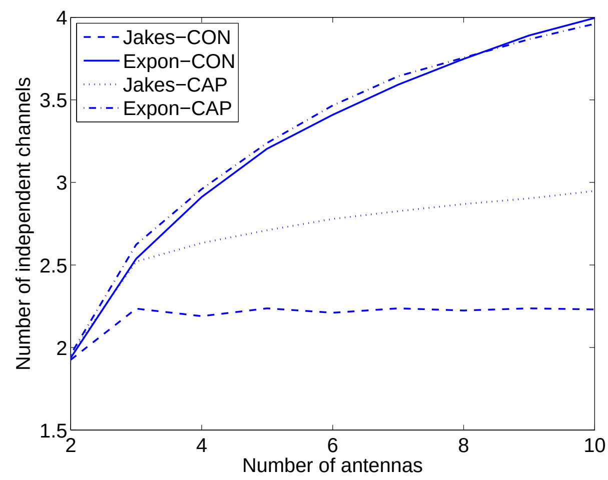

Fig. 8. NIC vs antenna numbers for $\mathrm{NIC_{CON}}$ and $\mathrm{NIC_{CAP}}$ with the exponential and Jakes correlation models. The corrected version of $\mathrm{NIC_{CAP}}$ is used at SNR $= 5$ dB and $d_{0.5} = 0.5\lambda$. Array length fixed at one wavelength; no coupling considered.

SNR, a direct comparison is not sensible as they have two different approaches to viewing NIC.

Finally, in Fig. 9 we consider the MIMO extensions to $\mathrm{NIC_{CAP}}$. Consider a $(2,2)$ MIMO system with $(\mathbf{R}_r)_{ij} = (\mathbf{R}_t)_{ij} = \rho^{|i-j|}$ for $i = 1,2$ and $j = 1,2$. Linearly interpolating the iid mean capacity and the iid capacity variance for the 4 systems (1,1), (1,2), (2,1) and (2,2) leads to curves of constant mean and constant variance which match the true mean and variance of the system. The mean is computed from (7) and the variance from the result in [27]. We seek the intersection of the two curves in the region $1 \leq N_r \leq 2$, $1 \leq N_t \leq 2$ since the correlation is expected to reduce the NIC. These curves are shown in Fig.9 for $\rho = 7$ dB and a decorrelation distance of 0.3 at both transmitter and receiver. Clearly such an intersection does not exist.

No results for the MIMO case are shown. For the construction approach, this is because the MIMO results follow

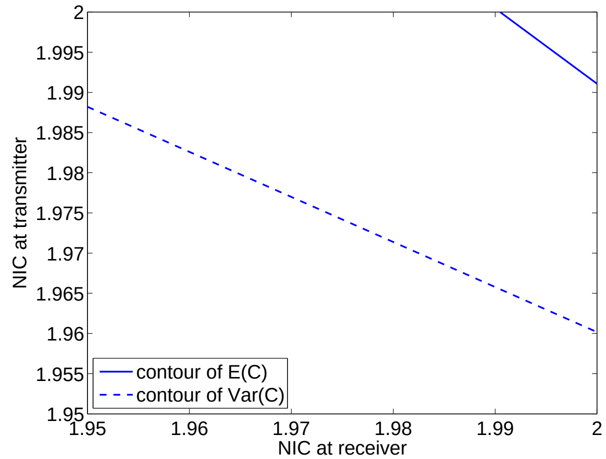

Fig. 9. Contours of equal mean capacity and equal capacity variance for a (2,2) system with $\rho = 7$ dB and a decorrelation distance of 0.3 λ at transmitter and receiver. Array length fixed at one wavelength; no coupling considered.

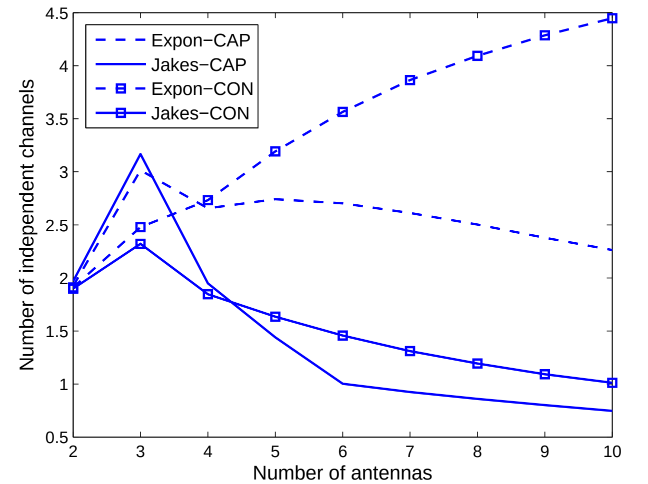

Fig. 10. NIC vs antenna numbers for NIC_{CON} and NIC_{CAP} with the exponential correlation model. The corrected version of NIC_{CAP} is used at SNR = 10 dB and $d_{0.5} = 0.5\lambda$. Array length fixed at one wavelength; effects of coupling considered.

from the SIMO case using (24) and assuming a Kronecker structure. For the capacity based approach, MIMO results are not reliable as discussed above.

All the results so far suggest that the construction approach is a robust solution. Originally developed for the Rayleigh fading SIMO channel it has been extended to Rician and MIMO channels. Also, it has retained sensible physical properties even in the presence of closely packed antennas. Hence, $\text{NIC}_{\text{CON,MIMO}}$ is proposed as a NIC measure for general channels. In such applications it loses its physical interpretation because it is based on the construction of Gaussian channels. Nevertheless, in the case of zero-mean channels it is solely based on the channel correlation matrix and hence it can be employed as an ad-hoc measure for any channel and any antenna configuration. For channels with a non-zero mean the approach used in the Rician case can be extended and so $\text{NIC}_{\text{CON,MIMO}}$ can be used for any channel model.

Finally, we consider the effects of mutual coupling in Fig. 10. Here, we investigate a SIMO system with an SNR of 10 dB and $d_{0.5} = 0.5\lambda$. Both NIC_{CAP} and NIC_{CON} are shown for Jakes and exponential correlation models. In contrast to Fig. 8, where no coupling was considered, the value of NIC has clearly been reduced by coupling. In all cases but one, the value of NIC increases initially but then decreases steadily as the number of antennas increases. Clearly, the physical effects of deploying antennas has an effect not inherent in the spatial correlation models. The trend for the exponential correlation model to yield higher NIC values remains the same when coupling is present. Given that NIC_{CAP} is able by construction to cater for the changes in received power created by coupling, it can be argued that here, the capacity based metric is preferable. On the other hand, NIC_{CON} is perhaps more flexible and in the absence of strong coupling it has several advantages.

VII. Conclusions

We considered the problem of quantifying the effective number of independent channels in a spatially correlated multiple antenna system. We discussed a number of previously proposed metrics and introduced two novel metrics to address their shortcomings.

We proposed a construction metric based on the Cholesky decomposition of the correlation matrix. The metric has a physical interpretation as the total power contained in the independent components of the channel. The metric returns a plausible value of NIC for a variety of correlation models considered, while highlighting the importance of antenna placement on the metric output. This metric is an attractive solution as it can be computed directly from the correlation matrix (and the LOS matrix in LOS channels), has simple extensions to MIMO systems and leads to simple insights into the effects of correlation and LOS strength.

The second metric introduced defines NIC as the number of receive antennas in an uncorrelated SIMO system of equal capacity. A correction factor was introduced to remove the effects of array gain from the capacity expression. For the exponential and Jakes correlation models the resulting metric was shown to be well behaved, showing no significant fluctuation with SNR and a moderate increase with the introduction of additional antennas to a fixed size array. However, extension to the MIMO case is problematic for NIC_{CAP} and it is inherently dependent on SNR. Nevertheless, this measure has certain advantages when mutual coupling is present.

Appendix A
Lower Bound on NIC_{CON}

Consider the correlation matrix, $\mathbf{R}_K$, for a SIMO system with K antennas and exponential correlation defined by the parameter, $\alpha = a^{-d}$. The Cholesky decomposition is denoted by $\mathbf{R}_K = \mathbf{A}_K\mathbf{A}_K^\dagger$. When $K = 1$, $\mathbf{R}_1 = \mathbf{A}_1 = 1$ and for $K = 2$,

$$\mathbf{R}_2 = \begin{pmatrix} 1 & \alpha \\ \alpha & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \alpha & \sqrt{1-\alpha^2} \end{pmatrix} \begin{pmatrix} 1 & \alpha \\ 0 & \sqrt{1-\alpha^2} \end{pmatrix}.$$

Hence, for $K = 1$ and $K = 2$, the diagonal of $\mathbf{A}_K$ is given by $(1 \ \sqrt{1-\alpha^2} \ \ldots \sqrt{1-\alpha^2})$ as desired. Now, assume that the diagonal has this form for some $K \geq 1$ and write

$$\mathbf{R}_{K+1} = \begin{pmatrix} \mathbf{R}_K & \mathbf{v}_{K+1}^T \\ \mathbf{v}_{K+1} & 1 \end{pmatrix},$$

where $\mathbf{v}_{K+1} = (\alpha^K \ \alpha^{K-1} \ldots \alpha)$. The Cholesky decomposition of $\mathbf{R}_{K+1}$ is given by

$$\begin{aligned} \mathbf{R}_{K+1} &= \mathbf{A}_{K+1}\mathbf{A}_{K+1}^{\dagger} \qquad (28) \\ &= \begin{pmatrix} \mathbf{A}_K & \mathbf{0} \\ \mathbf{r}_{K+1} & \beta \end{pmatrix} \begin{pmatrix} \mathbf{A}_K^T & \mathbf{r}_{K+1}^T \\ \mathbf{0}^T & \beta \end{pmatrix}. \end{aligned}$$

The diagonal of $\mathbf{A}_K$ has the desired form by the induction hypothesis. Hence, it suffices to show that $\beta = \sqrt{1-\alpha^2}$. Looking at the last column of $\mathbf{R}_{K+1}$ in (29) gives $\mathbf{A}_K\mathbf{r}_{K+1}^T = \mathbf{v}_{K+1}^T$ so that $\mathbf{r}_{K+1}^T = \mathbf{A}_K^{-1}\mathbf{v}_{K+1}^T$. We also have $\mathbf{R}_K = \mathbf{A}_K\mathbf{A}_K^{\dagger}$ so that $\mathbf{A}_K^T = \mathbf{A}_K^{-1}\mathbf{R}_K$. Now the last column of $\mathbf{R}_K$ is $\mathbf{v}_{K+1}^T/\alpha$ and denoting the last column of $\mathbf{A}_K^T$ by $(\mathbf{A}_K^T)_{\cdot K}$ gives $(\mathbf{A}_K^T)_{\cdot K} = \mathbf{A}_K^{-1}\mathbf{v}_{K+1}^T/\alpha = \mathbf{r}_{K+1}^T/\alpha$. Hence, from (29), $\mathbf{r}_{K+1}\mathbf{r}_{K+1}^T + \beta^2 = 1$ which implies $\alpha^2(\mathbf{A}_K^T)_{\cdot K}^T(\mathbf{A}_K^T)_{\cdot K} + \beta^2 = 1$. Rewriting gives $\alpha^2(\mathbf{R}_K)_{KK} + \beta^2 = 1$ and $\beta = \sqrt{1-\alpha^2}$ as desired, since $(\mathbf{R}_K)_{KK} = 1$.

APPENDIX B
DERIVATION OF $\mathrm{NIC}_{\mathrm{CON,MIMO}}$ FOR THE KRONECKER CASE

Since $\mathbf{H} = \mathbf{R}_r^{\frac{1}{2}}\mathbf{U}\mathbf{R}_t^{\frac{1}{2}}$ and $\mathbf{R} = E[\mathrm{vec}(\mathbf{H})\mathrm{vec}(\mathbf{H})^{\dagger}]$ it follows that $\mathbf{R} = \mathbf{R}_t \otimes \mathbf{R}_r$ [20]. Now, by the mixed-product property of Kronecker products [28] we have

$$\begin{aligned} \mathbf{R} &= \mathbf{A}^{(t)}\mathbf{A}^{(t)\dagger} \otimes \mathbf{A}^{(r)}\mathbf{A}^{(r)\dagger} \\ &= (\mathbf{A}^{(t)} \otimes \mathbf{A}^{(r)})(\mathbf{A}^{(t)\dagger} \otimes \mathbf{A}^{(r)\dagger}), \qquad (29) \end{aligned}$$

which shows that the Cholesky decomposition of $\mathbf{R}$ is $\mathbf{R} = \mathbf{A}\mathbf{A}^{\dagger}$, where $\mathbf{A} = \mathbf{A}^{(t)} \otimes \mathbf{A}^{(r)}$. Now, since $\mathbf{A}^{(t)}$ and $\mathbf{A}^{(r)}$ are both lower triangular, it follows that, as in (23),

$$\sum_{j=1}^{MN} |A_{jj}|^2 = \sum_{i=1}^{N} |A_{ii}^{(t)}|^2 \sum_{j=1}^{M} |A_{jj}^{(r)}|^2.$$

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